Quantum geometry and quantum algorithms

Quantum geometry and quantum algorithms

S Garnerone[1,2], A Marzuoli[3] and M Rasetti[1,2]

[1] Dipartimento di Fisica, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy
[2] Institute of Scientific Interchange, Villa Gualino, Viale Settimio Severo 75, 10131 Torino, Italy
[3] Dipartimento di Fisica Nucleare e Teorica, Universita' degli Studi di Pavia and Istituto
Nazionale di Fisica Nucleare, Sezione di Pavia, via A Bassi 6, 27100 Pavia, Italy

E-mail: silvano.garnerone@polito.it, annalisa.marzuoli@pv.infn.it and mario.rasetti@polito.it

Abstract
Motivated by algorithmic problems arising in quantum field theories whose
dynamical variables are geometric in nature, we provide a quantum algorithm
that efficiently approximates the coloured Jones polynomial. The construction
is based on the complete solution of the Chern–Simons topological quantum
field theory and its connection to Wess–Zumino–Witten conformal field theory.
The coloured Jones polynomial is expressed as the expectation value of the
evolution of the q-deformed spin-network quantum automaton. A quantum
circuit is constructed capable of simulating the automaton and hence of
computing such an expectation value. The latter is efficiently approximated
using a standard sampling procedure in quantum computation.

PACS numbers: 03.67.Lx, 02.10.Kn, 04.60.Kz, 04.60.Nc

1. Introduction

A new frontier of quantum information is the search for algorithms capable of addressing
problems in low-dimensional geometry and topology. The Jones polynomial [1] characterizes
the topology of knots and links (collections of circles in 3-space) and is associated with
the expectation value of a Wilson loop operator in quantum Chern–Simons field theory in
three dimensions. The algebraic content of this theory is encoded into a quantum group
structure. The Jones polynomial is the link invariant obtained with all the component knots
labelled with the fundamental irrep of the quantum deformation of $SU(2)$, denoted in the
following by $SU(2)_q$. Efficient quantum algorithms for approximating the Jones polynomial
have been recently proposed in [2]. In [3], we introduced the q-deformed spin-network
automaton model. The spin-network quantum simulator model, which essentially encodes the
(quantum deformed) $SU(2)$ Racah–Wigner tensor algebra, was shown [3] to be capable of
implementing families of finite-states and discrete-time quantum automata which accept the
language generated by the braid group, and whose transition amplitudes are indeed coloured

Jones polynomials. The latter are an extension of the Jones polynomial with arbitrary irreps of $SU(2)_q$ labelling the component knots. In this paper, we shall explicitly construct a quantum circuit which efficiently simulates the dynamics of these automata and hence, if appropriately sampled with a set of measurements, approximates the coloured Jones polynomial. We shall discuss the complexity of the circuit showing that, since the time complexity of the spin-network automaton is polynomial in the size of the input (depending on the index of the braid group and on the number of crossings of the knot diagram), the algorithm that efficiently simulates the automata also provides an efficient estimation of the link invariant.

The paper is organized as follows. In section 2 we briefly review the physical setting of quantum geometry and the role that in it play the topological invariants. In section 3 we concisely describe the structure of the spin-network quantum automaton. In section 4 we provide the details of the quantum algorithm that approximates the coloured Jones polynomials and of the corresponding quantum circuit. In section 5 we provide a few concluding remarks and discuss possible future developments and extensions of the methods and concepts introduced in the paper.

2. Quantum geometry and topological invariants

General relativity—the prototype of physical theories whose dynamical variable, the gravitational field, is geometric in nature—still represents a major improvement in the 'geometrization programme' stated by Klein and Einstein almost one century ago. These ideas laid dormant long after the birth of quantum mechanics and quantum field theory. In particular, the quest for a quantum gravity theory dates back to 1960s, when Arnowitt, Deser and Misner [4] introduced the so-called $(3 + 1)$ decomposition of Einstein field equations, a Hamiltonian reformulation of general relativity to be assumed as the basic ingredient for constructing a canonical quantization scheme for gravity. We refer the reader to the classical textbooks [5, 6] for accounts on quantum general relativistic theories up to 1970s.

Nowadays such approach has been almost abandoned in favour of (hopefully) more effective quantization schemes, but a number of substantial contributions developed in that golden age keep circulating. A good example is provided by Wheeler's 'geometrodynamics', which embodies the concept of 'quantum geometry' of the physical three-dimensional space, to be thought of as quantum fluctuations of (diffeomorphism classes of) 3-metrics within the 'superspace' [7]. As we shall see below, three-dimensional extended objects—more precisely, smooth 3-manifolds endowed with Riemannian or Lorentzian metric tensors—with their rich geometric structure play a prominent role in Chern–Simons quantum field theories and associated statistical field theories. Moreover, models of quantum gravity in three spacetime dimensions represent by themselves very useful toy models in view of generalizations to the physically significant four-dimensional case.

Euclidean quantum field theory is the quantization procedure of a classical field theory based on 'functional integration', over the space of quantum fluctuations of the physical fields $\{\phi\}$, of $\exp[-S(\{\phi\})/\hbar]$, where $S(\{\phi\})$ is the classical action defined in the Wick-rotated counterpart of Minkowskii spacetime [8]. This approach to quantization can be related to classical statistical field theory, and consequently it inherits the language and methods proper of statistical mechanics (partition functions, phase transitions, etc). This latter feature is particularly fruitful if some kind of discretization prescription is applied to the classical theory and suitably extended to the path integrals which turn out to be interpretable as statistical sums or partition functionals. Indeed, the most successful quantization scheme for general relativity, the 'sum over histories', was proposed by Hawking and Hartle [9] borrowing techniques from the Euclidean path integral approach mentioned above. Its discretized version, simplicial

quantum gravity, relies on Regge's discrete reformulation of classical general relativity [10] and has been widely addressed in the last two decades (see e.g. [11, 12] and references therein).

The geometrization programme referred to at the beginning of this section was in some sense rephrased as a 'gauge principle' by Yang and Mills in [13]. Non-Abelian gauge theories interacting with matter fields and their quantized counterparts still play a central role in the physics of fundamental interactions, while pure Yang–Mills theories (classical and quantum) were recognized to encode a number of interesting geometric features (see e.g. the reviews [14, 15]).

Within the class of quantum Yang–Mills theories we focus our attention on 'topological' quantum field theories (TQFT), formulated in terms of axioms by Atiyah in [16] (see also [17, 18]). Such theories—quantized through the path integral prescription starting from a classical Yang–Mills action defined on an orientable Riemannian D-dimensional space(time)— are characterized by gauge invariant partition functions and observables (correlation functions) depending only on the global structure of the space on which the theories live. The latter geometric functionals are computable by standard techniques in quantum field theory and provide novel representations of 'topological invariants' for D-manifolds (and/or for particular submanifols embedded in the ambient space) which are of prime interest both in mathematics and in theoretical physics. All of this enlightens a new kind of connection between geometry and quantum physics: in TQFT the physical degrees of freedom of spacetime geometry are global and not local. Four-dimensional Einstein gravity quantized through the Euclidean path integral is not a TQFT; however the role of quantum 3-geometry is once more enhanced since gravity in three spacetime dimensions can be reformulated as a gauge theory closely related to the $SU(2)$ Chern–Simons TQFT [19, 20].

Without entering into technical details on TQFT in general, let us just recall some of the basic ingredients of Chern–Simons quantum field theory.

The classical $SU(2)$ Chern–Simons action for the 3-sphere S^3 (the simplest compact, oriented 3-manifold without boundary) is given by

$$kS_{\text{CS}}(A) = \frac{k}{4\pi} \int_{S^3} \text{tr}\left(A dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{1}$$

where A is the connection 1-form with value in the Lie algebra $su(2)$ of the gauge group, k is the coupling constant, d is the exterior differential, $\wedge$ is the wedge product of differential forms and the trace is taken over Lie algebra indices. The partition function of the quantum theory is obtained from the 'path integral' prescription, by integrating the exponential of i times the classical action (1) over the space of gauge-invariant flat $SU(2)$ connections (the field variables) according to the formal expression

$$\mathbf{Z}_{\text{CS}}[S^3; k] = \int [DA] \exp\left\{\frac{ik}{4\pi} S_{\text{CS}}(A)\right\}, \tag{2}$$

where the coupling constant k is constrained to be a positive integer by the gauge-invariant quantization procedure. The generating functional (2), written for a generic compact oriented 3-manifold $\mathcal{M}^3$ with $\partial \mathcal{M}^3 = \emptyset$, is a global invariant, namely it depends only on the topological type of $\mathcal{M}^3$. This is basically due to the feature that the space of solutions of quantum CS theory is finite dimensional [19].

The gauge-invariant observables in the quantum CS theory are expectation values of Wilson line operators associated with oriented knots (or links) embedded in the 3-manifold (commonly referred to as Wilson 'loop' operators). Knots and links are 'coloured' with irreducible representations (irreps) of the gauge group $SU(2)$, restricted to values ranging over the set $\{0, 1/2, 1, 3/2, \ldots, k/2\}$. Integer k will be related to the deformation parameter q in $U_q(su(2))$, the deformed universal enveloping algebra of $SU(2)$, with $q = \exp\left(\frac{-2i\pi}{k+2}\right)$.

In particular, the Wilson loop operator associated with a knot K carrying a spin-j irreducible representation is defined, for a fixed root of unity q, as (the trace of) the holonomy of the connection 1-form A evaluated along the closed loop $K \subset S^3$, namely

$$\mathbf{W}_j[K; q] = \mathrm{tr}_j \, P \exp \oint_K A, \tag{3}$$

where P denotes path ordering.

For a link L made of a collection of knots $\{K_l | l = 1, \ldots, s\}$, each labelled by an irrep, the expression of the composite Wilson operator reads

$$\mathbf{W}_{j_1 j_2 \ldots j_s}[L; q] = \prod_{l=1}^{s} \mathbf{W}_{j_l}[K_l; q]. \tag{4}$$

In the framework of the path integral quantization procedure, expectation values of observables are defined as functional averages weighted with the exponential of the classical action. In particular, the functional average of the Wilson operator (4) is

$$\mathcal{E}_{j_1 \ldots j_s}[L; q] = \frac{\int [DA] \mathbf{W}_{j_1 \ldots j_s}[L] \exp^{\frac{ik}{4\pi} S_{CS}(A)}}{\int [DA] \exp^{\frac{ik}{4\pi} S_{CS}(A)}}, \tag{5}$$

where $S_{\mathrm{CS}}(A)$ is the CS action for the 3-sphere given in (1) and the generating functional in the denominator is usually normalized to 1. It can be shown that this expectation value, which essentially[4] coincides with the coloured Jones polynomial [21–23], depends only on the isotopy type of the oriented link L and on the set of irreps $\{j_1, \ldots, j_s\}$. The original Jones polynomial [1] is recovered when a spin-$\frac{1}{2}$ representation is placed on each link component. However, the coloured link invariants are more effective than Jones' in detecting knots, as discussed in [25].

The coloured invariants (5) are the basic objects that will be addressed for computational purposes in the rest of this paper. The reader interested in an account of their construction through the quantum group approach may refer to [26] (section 3), where the issue of (unitary) braid group representations is also considered. In the following section, we shall use yet another kind of approach [27], which relies on the introduction of the boundary Wess–Zumino–Witten conformal field theory into the Chern–Simons setting. Such approach provides a particularly useful presentation of the coloured Jones polynomials as expectation values of unitary braiding operators in WZW theory.

We leave for the concluding remarks at the end of the paper the discussion of possible extensions of the quantum algorithm discussed in the following session to the other hard problems arising in the theory of closed (hyperbolic) 3-manifolds.

3. The spin-network quantum automaton

In the first subsection of this section, we shall briefly review automata theory and define basic concepts of formal language theory. Then we describe the model of quantum automaton relevant in the present context: the *spin-network quantum automaton*, which provides a natural connection between quantum computation and link invariants [3].

---

[4] These polynomials are actually invariants of 'framed links', see e.g. [23, 24]. The connection between $\mathcal{E}_{j_1 \ldots j_s}[L; q]$ and the genuine coloured Jones polynomial is $J_{j_1 \ldots j_s}(L, q) = \{q^{-3w(L)/4}/(q^{1/2} - q^{-1/2})\} \mathcal{E}_{j_1 \ldots j_s}[L]$, once suitable normalizations for the unknots have been chosen. Here $w(L)$ is the writhe associated with the planar diagram $D(L)$ of the oriented link L, defined as $w(L) = \sum_p \varepsilon(p)$. The summation runs over the self-crossing points of $D(L)$ and $\varepsilon(p) = \pm 1$ according to simple combinatorial rules. The writhe is easily evaluated from the link diagram by simple counting arguments.

3.1. Automata theory

The theory of automata and formal languages addresses in a rigorous way the notions of computing machines and computational processes. We review first some of the basic concepts.

If $\mathcal{A}$ is an alphabet, made of letters, digits or other symbols, and $\mathcal{A}^*$ denotes the set of all finite sequences of words over $\mathcal{A}$, a language $\mathcal{L}$ over $\mathcal{A}$ is a subset of $\mathcal{A}^*$. The length of the word w is denoted by $|w|$ and w_i is its ith symbol. The concatenation of two words $u, v \in \mathcal{L}$ is denoted simply by uv. In 1950s Noam Chomsky [28] introduced a four-level hierarchy describing formal languages according to their structure (grammar and syntax): regular languages, context-free languages, context-sensitive languages and recursively enumerable languages. The processing of each language is inherently related to a particular computing model (see e.g. [29] for an account on formal languages). Here we are interested in finite-state automata, the machines able to accept regular languages.

A deterministic finite state automaton consists of a finite set of states S, an input alphabet $\mathcal{A}$, a transition function $F : S \times \mathcal{A} \to \text{S}$, an initial state s_{in} and a set of accepting states $S_{\text{acc}} \subset S$. The automaton starts in s_{in} and reads an input word w from left to right. At the ith step, if the automaton reads the symbol w_i, then it updates its state to $s' = F(s, w_i)$, where s is the state of the automaton reading w_i. One says that the word has been accepted if the final state reached after reading w is in S_{acc}.

In the case of a non-deterministic finite-state automaton, the transition function is defined as a map $F : S \times \mathcal{A} \to \mathcal{P}(\text{S})$, where $\mathcal{P}(\text{S})$ is the power set of S. After reading a particular symbol, the transition can lead to different states, according to some assigned probability distribution.

Generally speaking, quantum finite-state automata are obtained from their classical probabilistic counterparts by moving from the notion of (classical) probability, associated with transitions, to quantum probability amplitudes. Computation takes place inside the computational Hilbert space through unitary matrices. In the present context, we shall confine our attention to the so-called measure-once quantum automaton [30]. The latter is a 5-tuple $M = (Q, \Sigma, U, |\mathbf{q}_0\rangle, |\mathbf{q}_f\rangle)$, where Q is a finite set of quantum states, Σ is a finite input alphabet with an end-marker symbol # and $U(\Sigma) : Q \to Q$ is the set of transition functions induced by reading Σ. The probability amplitude for the transition from the state $|\mathbf{q}\rangle$ to the state $|\mathbf{q}'\rangle$ upon reading the symbol $\sigma \in \Sigma$ is therefore $\langle \mathbf{q}|U(\sigma)|\mathbf{q}'\rangle$. The state $|\mathbf{q}_0\rangle \in Q$ is the initial configuration of the system, and $|\mathbf{q}_f\rangle$ is an accepting final state. For all states and symbols the function $U(\sigma)$ must be represented by unitary operators. The end-marker # is the last symbol of each input word and computation terminates after reading it. At the end of the computation the configuration of the automaton is measured; if it is in an accepting state then the input is accepted, otherwise it is rejected. The probability amplitude for the automaton of accepting the string w is given by $f_M(w) = \langle \mathbf{q}_f|U(w)|\mathbf{q}_0\rangle$, $U(w) \equiv: \prod_{w_i \in \Sigma} U(w_i) :$ for $w =: \prod_i w_i :$ $(: \cdot :$ denotes ordered product, and for w we used the product symbol to denote concatenations). The explicit form of $f_M(w)$ defines the language $\mathcal{L}$ accepted by that particular automaton. If $\hat{P}$ denotes the projector over the accepting states, the probability for the automaton of accepting the string w is given by $p_M(w) = \|\hat{P}|q_w\rangle\|^2$, where $|q_w\rangle \equiv U(w)|\mathbf{q}_0\rangle$.

3.2. The q-deformed spin-network automaton

In this subsection, we review briefly the structure of the *q-deformed spin-network* automaton model, first discussed in [3]. This quantum automaton is an extension of the spin-network model of computation, introduced in [31] and worked out in [32], constructed on the combinatorics of the Racah–Wigner algebra of the quantum group $SU(2)_q$. The *q-deformed*
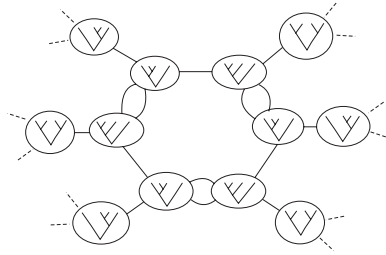
Figure 1. A portion of the spin-network graph. Unlabelled trees are associated with particular binary coupling schemes for the total Hilbert space. Single edges correspond to recoupling-like transformations, double edges correspond to braid-like transformations.
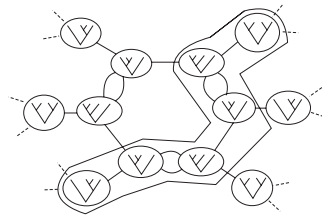

Figure 2. A portion of the spin-network graph with a ribbon denoting a path on the graph corresponding to a particular evolution of the spin-network automaton.

spin network is a model for a quantum automaton capable of processing the braid group language. From now on we shall refer to the model simply as *spin network*, subsuming the use of the deformed algebra.

The spin-network can be seen as a collection of graphs $\mathfrak{G}_n(V, E)$ parametrized by an integer n that is a measure of the size of the automaton. For fixed n, to each vertex $v \in V$ of $\mathfrak{G}_n(V, E)$ is associated the total Hilbert space $\mathfrak{H}_\flat^\mathbf{J}$ of the ordered tensor product of n irreps of $SU(2)_q$ (at q root of unity), together with a particular binary coupling scheme $\flat$ of the n angular momenta j_l ($l = 1, \ldots, n$) elements of the set $\mathbf{J}$. Different vertices correspond to different binary coupling schemes and admit a realization in terms of unrooted binary trees whose nodes are labelled with $SU(2)_q$ irreps. The edges $e \in E$ of $\mathfrak{G}_n(V, E)$ are associated with unitary evolutions connecting vertices (Hilbert spaces) belonging to V. A restriction is imposed on the type of allowed elementary unitary evolutions for the states in $\mathfrak{H}_\flat^\mathbf{J}$: they can be either *braid*-like or *recoupling*-like. The former are associated with a unitary representation of the braiding between two adjacent leaves of the binary tree; the latter are associated with reconfigurations of the binary coupling structure of the tree. The graph $\mathfrak{G}_n(V, E)$ is constructed in such a way that two vertices are connected by an edge if and only if there exists a *braid*-like or a *recoupling*-like unitary evolution mapping a state of the first vertex to a state of the second vertex (see figure 1). It was shown in [3] that it is possible to construct a finite-state quantum automaton able to process the language generated by the braid group B_n. Each graphical realization of the quantum automaton can be mapped onto a path in $\mathfrak{G}_n(V, E)$. The input word to the automaton is an element $b \in B_n$ and determines the evolution of the automaton according to its image $U(b)$, a unitary representation of B_n, which constitutes the transition rule. The evolution of the automaton is a sequence of allowed moves on $\mathfrak{G}_n(V, E)$, as depicted in figure 2.

The main result in [3] is that the probability amplitude for the automaton evolution associated with the unitary representation of a braid, whose closure is a particular link L, is equal to the coloured Jones polynomial of L. This result is based on the work of Kaul [27]. The details of the construction of the unitary representations of B_n will be summarized in the following section.

The connection between the spin-network computational model, the theory of quantum automata and link invariants will allow us to provide a quantum algorithm for the efficient approximation of topological invariants of knots. The advantage of using the spin-network quantum automaton resides in its transition rules, which can be straightforwardly expressed in the q-deformed co-algebra recoupling scheme. The latter naturally provides the set of unitary operations which are the building blocks of the quantum circuit evaluating the invariants. Previous discussion was aimed to mapping the problem of evaluating link invariants into the problem of simulating the corresponding evolution of the quantum automaton and considering henceforth the two problems as equivalent.

4. A quantum algorithm that approximates the coloured Jones polynomial

In this section, we provide a quantum algorithm that efficiently approximates the value of the coloured Jones polynomial. The interest in this problem stems from the fact that an additive approximation of the Jones polynomial is sufficient to simulate any polynomial quantum computation [36]. The construction of the algorithm involves three different contexts:

  (i) a topological context, where the problem is well defined and which allows us to recast the initial instance from the topological language of knot theory to the algebraic language of braid group theory;
 (ii) a field theoretic context, where tools from CS topological field theory and WZW conformal field theory are used to provide a unitary representation of the braid group;
(iii) a quantum information context, where the basic features of quantum computation are used to efficiently solve the original problem formulated in a field theoretic language.

We shall not discuss the topological context itself, where theorems and algorithms are available to relate links and braids, and refer the interested reader to [33, 34] and [26]. The field theoretic context will be discussed in the first subsection. The second subsection will deal with the basic structure of the algorithm and its computational complexity. In the last subsection, we shall complete the proof of efficiency and we shall provide notions needed to completely characterize the algorithm.

4.1. The Kaul construction

In [27], Kaul provides a unitary representation of the braid group and develops a method to evaluate observables in $SU(2)_q$ CS field theory on the 3-sphere S^3. His construction is based on the relationship between CS theory on a 3-manifold with boundary and the induced WZW conformal field theory on the boundary. Let us consider a 3-manifold $\mathcal{M}^3$ with a number n of two-dimensional boundaries $\Sigma^1, \Sigma^2, \ldots, \Sigma^n$. For each of these boundaries, say Σ^i, there are a number of Wilson lines carrying spins j_l^i intersecting the boundary at some 'puncture' P_l^i on the boundary (see figure 3).

We can associate to each Σ^i a Hilbert space $\mathfrak{H}^i$. The CS functional integral over $\mathcal{M}^3$ is then given as a state in the tensor product of such Hilbert spaces. Following the literature, in this section we shall henceforth denote by $SU(2)_k$ the quantum group $SU(2)_q$ with $q = \exp\left(\frac{2\pi i}{k+2}\right)$; in the following, we shall use both expressions interchangeably. The conformal blocks
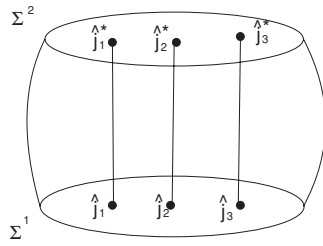
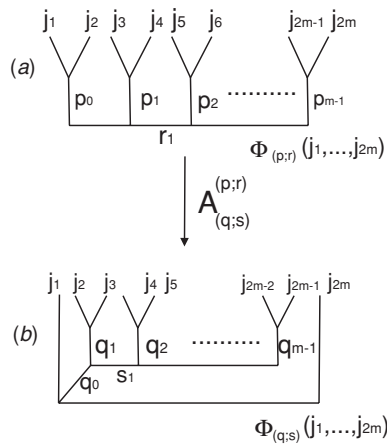**Figure 3.** Three Wilson lines intersecting the boundaries of a 3-sphere.

Figure 4. Duality transformation between two types of conformal blocks.

of $SU(2)_k$ WZW field theory on the boundaries Σ^i with punctures determine the properties of $\mathfrak{H}^i$. For each $\mathfrak{H}^i$ there are different bases related by duality of the correlators of the WZW conformal field theory.

These duality matrices can themselves be expressed in terms of q-deformed $SU(2)$ recoupling coefficients. This allows us to relate the unitaries generating the computational dynamics of the spin-network automaton to the observables of the CS field theory.

An important aspect in the construction developed in [27] is the close connection between links and braids. One obtains this important result by two main theorems. The first generalizes to coloured oriented braids a theorem, due to Birman [34], relating links to plats of braids. The second, which allows us to decompose the duality matrix associated with a general q-$3nj$ recoupling transformation into a sequence of elementary duality matrices associated with q-$6j$ recoupling transformations, reads the following theorem.

Theorem. *The correlators for 2m primary fields with spins $j_1, j_2, \ldots, j_{2m}$ in $SU(2)_k$ Wess–Zumino–Witten conformal field theory on S^2 are related to each other by*

$$|\Phi_{(\mathbf{p};\mathbf{r})}(j_1, \ldots, j_{2m})\rangle = \sum_{(\mathbf{q};\mathbf{s})} A_{(\mathbf{p};\mathbf{r})}^{(\mathbf{q};\mathbf{s})} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \\ \vdots & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix} |\Phi_{(\mathbf{q};\mathbf{s})}(j_1, \ldots, j_{2m})\rangle, \quad (6)$$
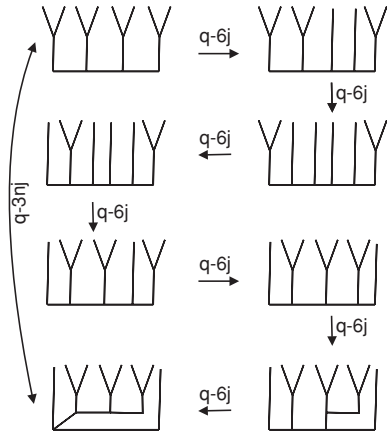
Figure 5. The sequence of decompositions into elementary recoupling transformations of a particular duality matrix.

where the duality matrix is given as a product of the basic duality coefficients for the four-point correlators as

$$A^{(\mathbf{q};\mathbf{s})}_{(\mathbf{p};\mathbf{r})} \begin{bmatrix} j_1 & j_2 \\ \vdots & \vdots \end{bmatrix} = \sum_{t_1,t_2,\ldots,t_{m-2}} \prod_{i=1}^{m-2} \left(A^{t_i}_{p_i} \begin{bmatrix} r_{i-1} & j_{2i+1} \\ j_{2i+2} & r_i \end{bmatrix} A^{s_{i-1}}_{t_i} \begin{bmatrix} t_{i-1} & q_i \\ s_i & j_{2m} \end{bmatrix} \right)$$

$$\times \prod_{l=0}^{m-2} A^{q_{l+1}}_{r_l} \begin{bmatrix} t_l & j_{2l+2} \\ j_{2l+3} & t_{l+1} \end{bmatrix}. \tag{7}$$

Here $r_0 \equiv p_0, r_{m-2} \equiv p_{m-1}, t_0 \equiv j_1, t_{m-1} \equiv j_{2m}, s_0 \equiv q_0, s_{m-2} \equiv q_{m-1}, \mathbf{j}_{2m} = \sum_{i=1}^{2m-1} \mathbf{j}_i$ and the spins meeting at trivalent vertices in figure 4 satisfy the fusion rules of the $SU(2)_k$ CFT. In figure 5, we provide a pictorial example of the content of the theorem.

The elements in the string $\{j_1, j_1^*, \ldots, {}_m, j_m^*\}$ will be referred to as **j**-type numbers, the elements in $\{p_0, \ldots, p_{m-1}\}$ as **p**-type numbers and the elements in $\{r_0, \ldots, r_{m-2}\}$ as **r**-type numbers.

A general n-strand coloured oriented braid is specified by giving n assignments $\hat{j}_i = (j_i, \epsilon_i)$, representing the spin and the orientation at each point on the upper and lower horizontal lines intersecting the strands. The generators of the groupoid of coloured oriented braids are

$$b_l \begin{pmatrix} \hat{j}^*_{l+1} & \hat{j}^*_l \\ \hat{j}_l & \hat{j}_{l+1} \end{pmatrix} \equiv b_l \begin{pmatrix} \hat{j}^*_1 & \cdots & \hat{j}^*_{l+1} & \hat{j}^*_l & \cdots & \hat{j}^*_n \\ \hat{j}_1 & \cdots & \hat{j}_l & \hat{j}_{l+1} & \cdots & \hat{j}_n \end{pmatrix}, \tag{8}$$

with $l \in \{1, \ldots, n-1\}$, where the '*' implies opposite orientation of the strand with respect to the horizontal line (figure 6).

The generators of coloured oriented braids satisfy the usual defining relations of the braid group (figure 7)

$$b_i \begin{pmatrix} \hat{j}^*_{i+1} & \hat{j}^*_i \\ \hat{j}_i & \hat{j}_{i+1} \end{pmatrix} b_{i+1} \begin{pmatrix} \hat{j}^*_{i+2} & \hat{j}^*_i \\ \hat{j}_i & \hat{j}_{i+2} \end{pmatrix} b_i \begin{pmatrix} \hat{j}^*_{i+2} & \hat{j}^*_{i+1} \\ \hat{j}_{i+1} & \hat{j}_{i+2} \end{pmatrix}$$

$$= b_{i+1} \begin{pmatrix} \hat{j}^*_{i+2} & \hat{j}^*_{i+1} \\ \hat{j}_{i+1} & \hat{j}_{i+2} \end{pmatrix} b_i \begin{pmatrix} \hat{j}^*_{i+2} & \hat{j}^*_i \\ \hat{j}_i & \hat{j}_{i+2} \end{pmatrix} b_{i+1} \begin{pmatrix} \hat{j}^*_{i+1} & \hat{j}^*_i \\ \hat{j}_i & \hat{j}_{i+1} \end{pmatrix},$$
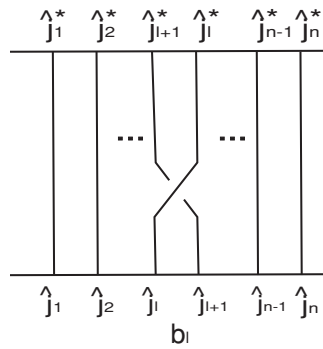
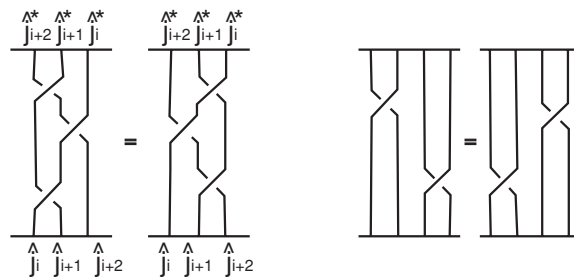**Figure 6.** A graphic realization of a generator of the coloured braid group.

Figure 7. Defining relations for the coloured braid group generators.

for $i = 1, \ldots, n-1$ and

$$b_i \begin{pmatrix} \hat{j}_{i+1}^* & \hat{j}_i^* \\ \hat{j}_i & \hat{j}_{i+1} \end{pmatrix} b_l \begin{pmatrix} \hat{j}_{l+1}^* & \hat{j}_l^* \\ \hat{j}_l & \hat{j}_{l+1} \end{pmatrix} = b_l \begin{pmatrix} \hat{j}_{l+1}^* & \hat{j}_l^* \\ \hat{j}_l & \hat{j}_{l+1} \end{pmatrix} b_i \begin{pmatrix} \hat{j}_{i+1}^* & \hat{j}_i^* \\ \hat{j}_i & \hat{j}_{i+1} \end{pmatrix},$$

for $|i - l| \geqslant 2$.

The *platting* of a coloured oriented braid on an even number of strands is the pairwise joining of contiguous strands, both from above and below. Birman's theorem, which relates oriented links to plats of ordinary braids [34], is extended in [27] to coloured oriented braids in such a way that a coloured oriented link is represented by the plat closure of an oriented coloured braid $b \begin{pmatrix} \hat{l}_1 & \hat{l}_1^* & \ldots & \hat{l}_m & \hat{l}_m^* \\ \hat{j}_1 & \hat{j}_1^* & \ldots & \hat{j}_m & \hat{j}_m^* \end{pmatrix}$, see figure 8.

We can finally describe now a method for evaluating the expectation value of an arbitrary Wilson link operator. Consider the 3-sphere S^3 with two three-balls removed. This is a manifold with two boundaries with the topology of the 2-sphere S^2. Let us place in this manifold $2m$ Wilson lines with spins $j_1, j_2, \ldots, j_{2m}$, such that all the spins generate an $SU(2)_q$ singlet connecting one boundary to the other. It is easily recognized that with these Wilson lines, we can realize any element of B_{2m} (see figure 9).

The CS functional integral over the 3-manifold can be realized by a state in the tensor product of vector spaces $\mathfrak{H}^1 \otimes \mathfrak{H}^2$, associated with the two boundaries Σ^1 and Σ^2. Conformal blocks can be chosen as basis vectors for these vector spaces. The inner products of these basis vectors are normalized according to

$$\langle \Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_1^*, \hat{j}_2^*, \ldots, \hat{j}_{2m}^*) | \Phi_{(\mathbf{u};\mathbf{v})}(\hat{j}_1, \hat{j}_2, \ldots, \hat{j}_{2m}) \rangle = \delta_{\mathbf{p},\mathbf{u}} \delta_{\mathbf{r},\mathbf{v}}. \tag{9}$$
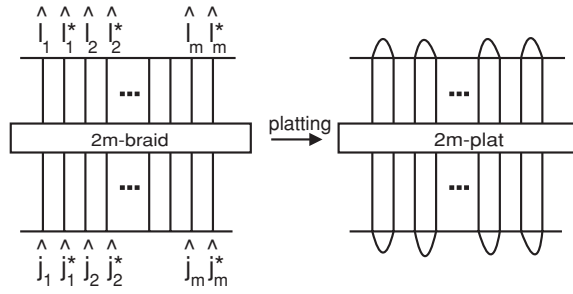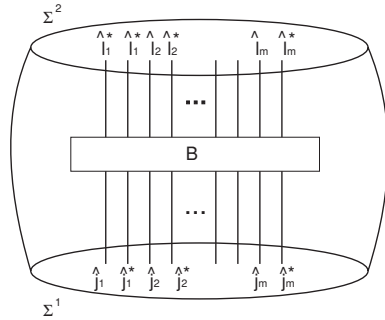
Figure 8. Platting of 2m coloured strands.

Figure 9. An arbitrary coloured braid pattern embedded into a 3-manifold with boundaries, Σ^1, Σ^2, with the topology of S^2.

The basis vectors $|\Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m})\rangle$ of the conformal blocks $\Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_1 \hat{j}_2 \ldots \hat{j}_{2m})$ are eigenfunctions of the odd indexed braiding generators b_{2l+1} defined in (8). The even indexed braid generators b_{2l} are diagonalized in the basis $|\Phi_{(\mathbf{q};\mathbf{s})}(\hat{j}_1, \ldots, \hat{j}_{2m})\rangle$. The following eigenvalue equations hold

$$\hat{b}_{2l+1}|\Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_{2l+1}, \hat{j}_{2l+2})\rangle = \lambda_{p_l}(\hat{j}_{2l+1}, \hat{j}_{2l+2})|\Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_{2l+2}, \hat{j}_{2l+1})\rangle, \tag{10}$$

$$\hat{b}_{2l}|\Phi_{(\mathbf{q};\mathbf{s})}(\hat{j}_{2l}, \hat{j}_{2l+1})\rangle = \lambda_{q_l}(\hat{j}_{2l}, \hat{j}_{2l+1})|\Phi_{(\mathbf{q};\mathbf{s})}(\hat{j}_{2l+1}, \hat{j}_{2l})\rangle. \tag{11}$$

Here $|\Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_l, \hat{j}_{l+1})\rangle \equiv |\Phi_{(\mathbf{p};\mathbf{r})}(\hat{j}_1, \ldots, \hat{j}_l, \hat{j}_{l+1}, \ldots, \hat{j}_{2m})\rangle$. The eigenvalues of the braiding matrices depend on the relative orientation of the strands, and for right-handed half twists (i.e. over-crossings) their value is

$$\lambda_t(\hat{j}, \hat{i}) \equiv (-)^{j+i-t} q^{(c_j+c_i)/2+c_{min(i,j)}-c_t/2}, \tag{12}$$

for parallel oriented strands, and

$$\lambda_t(\hat{j}, \hat{i}) \equiv (-)^{|j-i|-t} q^{-|c_j-c_i|/2+c_t/2}, \tag{13}$$

if the orientation is anti-parallel. Here c_j is the quadratic Casimir operator equal to $j(j+1)$ for the spin j representation. The eigenvalues (12) and (13) derive from the monodromy properties of the conformal blocks of the corresponding CFT. The associated unitary representation of the braid group is provided by the following theorem.

Theorem. *A class of representations* $\mathbf{K} : B_n \to U(d)$ *from the generators of the groupoid of coloured oriented braids into the unitary $d \times d$ matrices ($d = d(n, |b|)$) in the basis $|\Phi_{(\mathbf{p};\mathbf{r})}\rangle$, is given by*

$$\mathbf{K}\left[b_{2l+1}\begin{pmatrix}\hat{j}_{2l+2}^{*} & \hat{j}_{2l+1}^{*} \\ \hat{j}_{2l+1} & \hat{j}_{2l+2}\end{pmatrix}\right]_{(\mathbf{p};\mathbf{r})}^{(\mathbf{p}';\mathbf{r}')} = \lambda_{p_l}(\hat{j}_{2l+1}, \hat{j}_{2l+2})\delta_{\mathbf{p}}^{\mathbf{p}'}\delta_{\mathbf{r}}^{\mathbf{r}'}, \tag{14}$$

and by

$$\mathbf{K}\left[b_{2l}\begin{pmatrix}\hat{j}_{2l+1}^{*} & \hat{j}_{2l}^{*} \\ \hat{j}_{2l} & \hat{j}_{2l+1}\end{pmatrix}\right]_{(\mathbf{p};\mathbf{r})}^{(\mathbf{p}';\mathbf{r}')}$$

$$= \sum_{(\mathbf{q};\mathbf{s})} A_{(\mathbf{p};\mathbf{r})}^{(\mathbf{q};\mathbf{s})}\begin{bmatrix}\vdots & \vdots \\ j_{2l-1} & j_{2l+1} \\ j_{2l} & j_{2l+2} \\ \vdots & \vdots\end{bmatrix}\lambda_{q_l}(\hat{j}_{2l}, \hat{j}_{2l+1})A_{(\mathbf{q};\mathbf{s})}^{(\mathbf{p}';\mathbf{r}')}\begin{bmatrix}\vdots & \vdots \\ j_{2l-1} & j_{2l} \\ j_{2l+1} & j_{2l+2} \\ \vdots & \vdots\end{bmatrix}. \tag{15}$$

The proof that the defining relations for the braid generators are indeed satisfied can be found in [27].

Consequently, the expectation value of a Wilson loop operator for an arbitrary link L presented as a plat closure of a coloured oriented braid $b\binom{\hat{l}_1 \hat{l}_1^* \dots \hat{l}_m \hat{l}_m^*}{\hat{j}_1 \hat{j}_1^* \dots \hat{j}_m \hat{j}_m^*}$, generated by a word given in terms of the braid generators, is given by

$$V[L; \mathbf{j}; q] = \prod_{i=1}^{m}[2j_i + 1]\langle\Phi_{(\mathbf{0};\mathbf{0})}(\hat{l}_1, \dots, \hat{l}_m^*)|\mathbf{K}\left[b\begin{pmatrix}\hat{l}_1 & \cdots & \hat{l}_m^* \\ \hat{j}_1 & \cdots & \hat{j}_m^*\end{pmatrix}\right]|\Phi_{(\mathbf{0};\mathbf{0})}(\hat{j}_1, \dots, \hat{j}_m^*)\rangle, \tag{16}$$

where the multi-index $(\mathbf{0}; \mathbf{0})$ denotes the case in which all the elements in the set of $\mathbf{p}$ and $\mathbf{r}$-type numbers are equal to 0, while

$$[x] \doteq \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

is the standard notation for the quantum integer. The latter theorem gives us the explicit evaluation of the coloured polynomial. It can be shown that the Jones polynomial corresponds to a spin-$\frac{1}{2}$ representation living on all the components of the link.

4.2. The qubit representation

In this section, we show how to efficiently implement on a qubit-register the Kaul unitary representation $\mathbf{K}$ of the coloured braid group. We prove that each unitary matrix of $\mathbf{K}(B_{2m})$, interpreted as a gate acting on a qubit-register, can be efficiently decomposed into a set of universal elementary gates. This is done first encoding into a qubit-register the basis vectors used in $\mathbf{K}$ and then showing how $\mathbf{K}(b_i)$ can be efficiently compiled for every $b_i \in B_{2m}$.

Each vector in the basis set $\{|\Phi_{(\mathbf{p};\mathbf{r})}(j_1, j_1^*, \dots, j_m, j_m^*)\rangle\}$, corresponding to the conformal block shown in figure 4(*a*), is completely characterized by three sets of quantum numbers, $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{j}$, fully labeling the irreps of $SU(2)_q$. Recall that the $\mathbf{p}$-, $\mathbf{r}$- and $\mathbf{j}$-type numbers belong to the set $\{0, \frac{1}{2}, \dots, \frac{k}{2}\}$, where k is the Chern–Simons coupling constant. This means that each type of number can be specified using $\lceil\log_2(k + 1)\rceil$ qubits, where $\lceil r \rceil$ denotes the least integer $\geqslant r$. An element of the basis can then be encoded using $4m - 3) \times \lceil\log_2(k + 1)\rceil$ qubits. The register we need to use has to code only for the $\mathbf{p}$-type and $\mathbf{r}$-type numbers, implying that
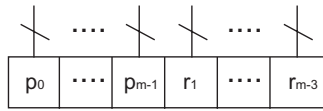
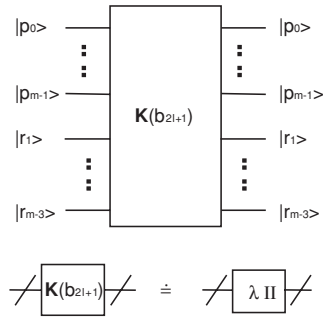**Figure 10.** Register of qubits for the Kaul representation.

Figure 11. The gate realization of the odd-indexed braid generators.
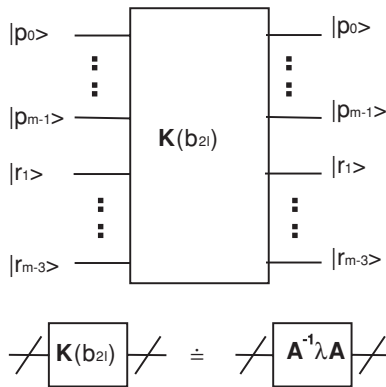

Figure 12. The gate realization of the even-indexed braid generators.

only $(2m - 3) \times \lceil \log_2(k + 1) \rceil$ qubits are sufficient. On the qubit-register we chose the order shown in figure 10.

The odd-indexed braid generators are diagonal matrices in the basis of the **K**-module, therefore there is no problem in implementing their action on the quantum register (see figure 11). The even-indexed braid generators have a less trivial representation (see figure 12).

Resorting to the representation in (15), we need to apply two duality matrices, or recoupling transformations, in order to explicitly construct the image of these generators under **K**. Each recoupling transformation can in turn be decomposed into a series of elementary *quantum* $6j$ transformations using (7), see e.g. figure 13.

It follows that the problem of efficiently compiling the general recoupling transformation from the eigenspace of odd-indexed braiding operations to the eigenspace of even-indexed braiding operations can be mapped into the easier problem of efficiently compiling a single $q\text{-}6j$
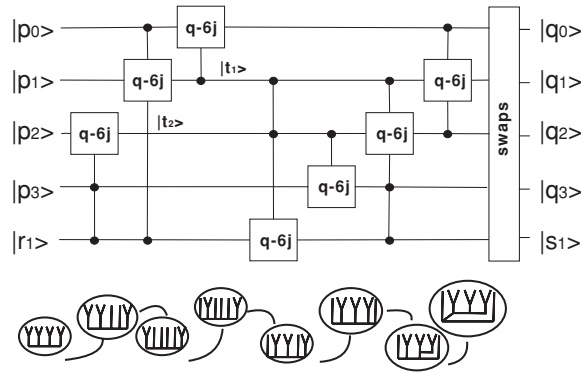
Figure 13. The quantum circuit implementing the decomposition of the eight-point conformal block in terms of q-$6j$ gates. The corresponding path on the spin-network graph is shown.
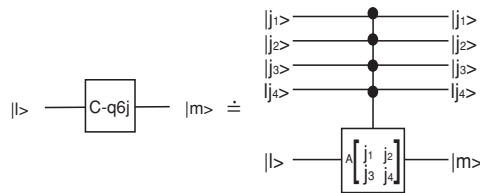

Figure 14. Definition of the controlled q-$6j$ transformation.
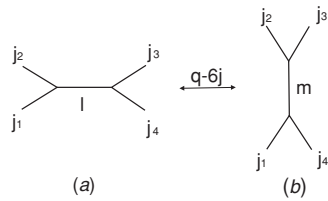

Figure 15. The q-$6j$ transformation.

transformation (figure 14). To this end, note that a q-$6j$ transformation, or the corresponding duality matrix, is a unitary transformation from states in the conformal block of figure 15(a) into states of the conformal block of figure 15(b).

Each element of the associated unitary matrix is defined in terms of the q-Racah coefficients by the following expressions:

$$|m\rangle\langle l|_{\mathbf{j}} \equiv A_m^l \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = (-)^{(j_1+j_2+j_3+j_4)}\sqrt{[2m+1][2l+1]} \begin{pmatrix} j_1 & j_2 & l \\ j_3 & j_4 & m \end{pmatrix}_q, \tag{17}$$

where all the relevant triplets of $SU(2)_q$ irreps satisfy the fusion rules of the WZW CFT. Recall that an explicit expression for the q-Racah coefficient [27] is

$$\begin{pmatrix} j_1 & j_2 & l \\ j_3 & j_4 & m \end{pmatrix}_q = \Delta(j_1,\dots,l)\Delta(j_3,\dots,l)\Delta(j_1,\dots,m)\Delta(j_2,\dots,m)$$
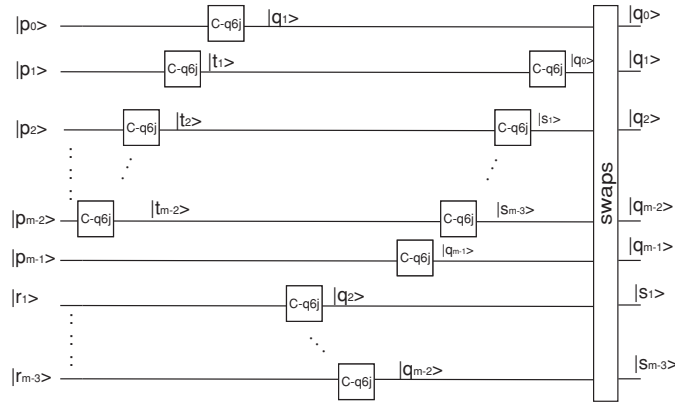
Figure 16. The quantum circuit implementing the general duality transformation.

$$\times \sum_{x \geqslant 0} (-)^x [x+1]! \, \{[x-j_1-j_2-l]![x-j_3-j_4-l]![x-j_1-j_4-m]!$$
$$\times [x-j_2-j_3-m]![j_1+j_2+j_3+j_4-x]![j_1+j_3+l+m-x]!$$
$$\times [j_2+j_4+l+m-x]!\}^{-1}, \tag{18}$$

where $[x]! \doteq [x][x-1]!$ with $[0]! = 1$, and $[\cdot]$ denotes the q integer. The sum is restricted to all allowed values of x such that the quantum integers entering the factorials are non-negative and

$$\Delta(a,b,c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}}.$$

Due to the finiteness of the sum, the coefficients (17) can thus be efficiently evaluated classically for all the $SU(2)_q$ irreps.

For what concerns the action on the qubit-register, elements (17) belong to unitary matrices of rank $2^{\lceil \log(k+1) \rceil}$, parametrized by the set **j** of those quantum numbers which remain unchanged along the transformation. The crucial fact to note here is that the dimension of these matrices is independent of the size of our problem, given by the index of the braid group and the number of crossings. Since there exist efficient methods to approximate unitary matrices of given dimension [35], there exists a sequence of universal gates that efficiently approximates every q-$6j$ transformation. The number of elementary q-$6j$ transformations needed to decompose a general q-$3nj$ recoupling transformation is $2m-3$, linear in the size of the problem. In conclusion, the Kaul representation **K** associated with an arbitrary coloured oriented braid can indeed be efficiently compiled on a standard quantum computer. The circuit implementing the decomposition of **K** is shown in figure 16.

4.3. The algorithm

The general structure of the quantum automaton whose dynamical evolution derived in [3] is characterized by probability amplitudes whose values correspond to observables of the CS QFT (coloured Jones polynomial), can be finally translated into an efficient quantum circuit by resorting to a procedure similar to that adopted by Aharonov, Jones and Landau in [2].

In section 4.2, we have shown that every unitary matrix belonging to the image of representation **K** is efficiently decomposable into a set of universal gates. We now prove that the information contained in the expectation value $V[L; \mathbf{j}; q]$ given by

$$\langle \Phi_{(\mathbf{0};\mathbf{0})}(\hat{l}_1, \ldots, \hat{l}_m^*) | \mathbf{K} \left[b \begin{pmatrix} \hat{l}_1 & \cdots & \hat{l}_m^* \\ \hat{j}_1 & \cdots & \hat{j}_m^* \end{pmatrix} \right] | \Phi_{(\mathbf{0};\mathbf{0})}(\hat{j}_1, \ldots, \hat{j}_m^*) \rangle$$

can be efficiently accessed by a series of measurements.

To begin with, we recall that the standard procedure used in quantum computation to evaluate the expectation value of a unitary relies on a scheme dubbed *Hadamard's trick*. The latter was applied for the first time in [2] dealing just with the problem of evaluating the Jones polynomial. We further recall that the notion of approximation used in the present context, formalized in [36], is that of *additive approximation*, which has the following meaning: given a normalized function $f(x)$, where x denotes an instance of the problem in the selected coding, we have an additive approximation of its value for each instance x if we can associate to $f(x)$ a random variable Z such that

$$\Pr\{|f(x) - Z| \leqslant \delta\} \geqslant 3/4,$$

for any $\delta \geqslant 0$. The time needed to achieve the approximation must be polynomial in the size of the problem and in δ^{-1}. The additive characterization of this approximation scheme underlies the fact that the interval $Z - \delta, Z + \delta]$, which we want to determine, is constructed adding $\pm\delta$ to Z. It also distinguishes this approximation scheme from the standard fully polynomial randomized approximation scheme. The normalization adopted for the coloured Jones polynomial of a link L is provided by the product, over all the link components, of the quantum integer related to the dimension of the $SU(2)_q$ irreps labeling the knots. The problem we are interested in can now be stated as follows.

Problem: Approximate coloured Jones polynomials (V_L). *Given a coloured braid $b \in B_{2m}$ of length ℓ, a colouring $\mathbf{c}$, a positive integer k and a real $\delta > 0$, we want to sample from a random variable Z which is an additive approximation of the absolute value of the coloured Jones polynomial of the plat closure of b, evaluated at $q = \exp\left(\frac{2\pi i}{k+2}\right)$, such that the following condition holds true*

$$\Pr(|V(L; \mathbf{j}; q) - Z| \leqslant \delta) \geqslant 3/4.$$

Here the colouring $\mathbf{c}$ denotes the set of all possibly different irreps of $SU(2)_q$ labeling the component knots of L.

In the following, we shall provide an efficient quantum algorithm for $\mathbf{V_L}$, which solves it in $O(\text{poly}(\ell, \delta^{-1}))$ steps. As in [2, 37], we need the following two lemmas in order to prove the efficiency of the algorithm.

Lemma 1. *Given a quantum circuit U of length $O(\text{poly}(n))$, acting on n qubits, and given a pure state $|\Phi\rangle$ which can be prepared in time $O(\text{poly}(n))$, then it is possible to sample in $O(\text{poly}(n))$ time from two random variables a and b, valued in $\mathbb{Z}_2$, in such a way that $\langle a + ib \rangle = \langle \Phi | U | \Phi \rangle$.*

Lemma 2. *For a sufficiently large N, given a set of random variables $\{r_i | i = 1, \ldots, N\}$ of average value m and square variance v*

$$\Pr\left(\left| N^{-1} \sum_{i=1}^{N} r_i - m \right| \geqslant \delta \right) \leqslant 2\exp(-N\delta^2/(4v)).$$
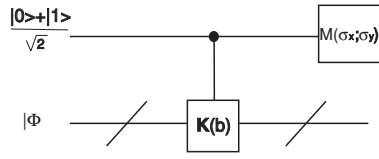
Figure 17. The circuit realizing the Hadamard's trick for the Kaul representation of the braid b. $M(\sigma_x; \sigma_y)$ denotes quantum measurement of either σ_x or σ_y.

The first lemma, which is essentially a reformulation of the Hadamard's trick, can be proved as follows. Introduce a single-qubit ancilla $\mathfrak{A}$ and denote by $\mathfrak{G}$ the Hilbert space of the qubits acted on by U. Define the unitary $C : \mathfrak{A} \otimes \mathfrak{G} \to \mathfrak{A} \otimes \mathfrak{G}$ through the action:

$$C(|0\rangle \otimes |\Phi\rangle) = |0\rangle \otimes |\Phi\rangle,$$
$$C(|1\rangle \otimes |\Phi\rangle) = |1\rangle \otimes (U|\Phi\rangle).$$

Initialize then the ancillary qubit in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \in \mathfrak{A}$ and prepare the system in the initial state $|\Phi\rangle$. The action of C maps the initial state into $|\Psi\rangle \in \mathfrak{A} \otimes \mathfrak{G}$

$$|\Psi\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle|\Phi\rangle + |1\rangle(U|\Phi\rangle)).$$

The reduced density matrix $\rho^{\mathfrak{A}}$ of the ancilla is thus equal to

$$\rho^{\mathfrak{A}} = \mathrm{Tr}_{\mathfrak{G}} |\Psi\rangle\langle\Psi| = \frac{1}{2} \mathrm{Tr}_{\mathfrak{G}} \begin{pmatrix} \Phi & \Phi U^{\dagger} \\ U\Phi & U\Phi U^{\dagger} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \langle\Phi|U^{\dagger}|\Phi\rangle \\ \langle\Phi|U|\Phi\rangle & 1 \end{pmatrix}$$

$$= \frac{1}{2}(\mathbb{I}_2 + \sigma_x \, \mathrm{Re}\langle\Phi|U|\Phi\rangle + \sigma_y \, \mathrm{Im}\langle\Phi|U|\Phi\rangle),$$

where Φ denotes the density matrix $|\Phi\rangle\langle\Phi|$ and σ_x, σ_y are Pauli matrices.

The mean value of a sequence of measurements of σ_x will approach $\mathrm{Re}\langle\Phi|U|\Phi\rangle$, whereas the mean value of a sequence of measurements of σ_y will approach $\mathrm{Im}\langle\Phi|U|\Phi\rangle$.

The second lemma, which is a modified version of the well known Chernoff bound, ensures us that we can approximate these values polynomially in the number N of samplings and in the inverse of the precision δ^{-1}.

Summarizing the qubit model for the Kaul representation can be used to efficiently compile a unitary representation of the coloured braid group, and a sampling procedure can then be used to efficiently estimate the value of the coloured Jones polynomial. The circuit that realizes all these steps is schematically depicted in figure 17.

In conclusion, the sampling lemma tells us that measurements of σ_x on the first qubit will provide the value for $\mathrm{Re}(V(L, \mathbf{j}, q))$, while measurements of σ_y on the first qubit will provide the value for $\mathrm{Im}(V(L, \mathbf{j}, q))$.

5. Conclusions

The q-deformed spin-network model provides the natural setting for a quantum automaton capable of processing the braid group language. Coding of information in the spin network is done in the frame of the coupling scheme associated with the 'co-power' $\Delta^n(SU(2)_q)$ (iterated co-product) of the network q-algebra. Such *parenthesized* coding lends itself quite naturally to deal with a number of hard combinatorial problems, ranging from finite groups word or isomorphism problems [38, 39] to the evaluation of topological invariants. Work is in progress on problems of the former type. Focusing mainly on the latter, for consistency with our introductory physical setting, we discuss here briefly the role played

by the coloured link polynomial introduced above in three-dimensional geometric topology [40, 41] in view of Thurston's 'geometrization programme' [42]. Indeed the possible extension of our quantum algorithm to address the (classically computationally hard) problems outlined below would represent a major breakthrough both in quantum computation and in the theory of closed (hyperbolic) 3-manifolds. On the physical side, such an achievement would open the possibility of 'controlling' the quantum algorithmic complexity of three-dimensional quantum gravity models. It is worth recalling here the alternative point of view introduced by the recent attempt by Lloyd of unifying quantum mechanics and gravity, where the very geometry of spacetime is a construct derived from the underlying quantum computation [43].

5.1. Reshetikhin–Turaev quantum invariants of 3-manifolds and their quantum complexity

At its foundation, knot theory is a branch of geometric topology, since it allows us to explore three-dimensional spaces by 'knotting' phenomena, namely embedded knots 'interact' with the topological structure of the ambient 3-manifold $\mathcal{M}^3$. The content of the latter remark is made more stringent by a theorem which asserts that every closed connected orientable 3-manifold can be obtained by Dehn 'surgery' along a framed link embedded in the 3-sphere S^3 (we refer to [23] for definitions and proofs). Roughly speaking, a tubular neighbourhood of each component of the embedded link L, represented by $S^1 \times D^2$ (D^2 being the 2-disk), is removed and replaced by $D^2 \times S^1$ in a suitable way, generating the new manifold. Formally

$$(S^3, L) \longrightarrow \mathcal{M}_L^3 \doteq S^3 \backslash L. \tag{19}$$

It can also be shown that equivalent links, namely links which are ambient isotopic, give rise to the same type of 3-manifolds (the manifolds obtained by surgeries in the 3-sphere along equivalent links are homeomorphic).

   The idea that the Jones polynomial at a root of unity q can be 'amplified' to achieve a 3-manifold quantum invariant dates back to Witten and was further implemented by a number of authors ([23] and references therein). Such invariants, which correspond to the partition function (2) evaluated for a manifold $\mathcal{M}_L^3$, are linear sums of Jones polynomials of copies of the link with the components replaced by various parallels of the original components. The authors of [44] propose to address the problem of designing quantum algorithms for Witten invariants by resorting to Temperley–Lieb algebra techniques.

   The quantum algorithm for the coloured Jones polynomials discussed in section 4, allows us to conjecture that the associated coloured 3-manifold quantum invariants at a fixed root of unity can be actually evaluated in a quite straightforward way. The explicit expression of the (Witten–)Reshetikhin–Turaev quantum invariant for a 3-manifold $\mathcal{M}_L^3$ to be used for computational purposes was proposed by Kirby and Melvin [22] and reads

$$\tau\left(\mathcal{M}_L^3; q\right) = \alpha_L \sum_{\mathbf{j}} [\mathbf{j}] \mathcal{E}_{j_1 \dots j_s}[L; q],$$

where $\mathbf{j}$ stands for the collective assignment of colourings to the link components, the summation is over all admissible colourings and $[\mathbf{j}] = \prod_{i=1}^s [2j_i + 1]$. $\mathcal{E}_{j_1 \dots j_s}[L; q]$ is given in (5) and $\alpha_L = b^{n_L} c^{\sigma_L}$. Here b and c are numbers depending on the integer k ($b = \sqrt{(2/k)} \sin \frac{\pi}{k}$ and $c = \exp[-2\pi \mathrm{i}(k-2)/8k]$), n_L is the number of link components and σ_L is the signature of the linking matrix of L. The linking matrix M_L of a framed link L is a symmetric matrix whose entry $(M_L)_{ij}$ for $i \neq j$ is the linking number between components i and j of L. The diagonal elements of M_L are defined to be the integers that give the framing of the individual components. The linking matrix , defined here in combinatorial terms, is related to the topology of $\mathcal{M}_L^3$ because its determinant (if it is non-zero) is the order of the first homology group of the manifolds.

5.2. The volume conjecture for hyperbolic 3-manifolds

In the framework of Thurston's geometrization program [42] (all three-dimensional manifolds can be reconstructed starting from eight types of model geometries), hyperbolic 3-manifolds play a special role. Recall that a 3-manifold is hyperbolic if it is endowed with a complete Riemannian metric with constant negative sectional curvature. The most interesting case is that of complete hyperbolic manifolds with finite volume (the volume being evaluated in the given metric) since the Mostow rigidity theorem asserts that any two such manifolds are homeomorphic if and only if they are homotopically equivalent (we refer to [45] for details and original references). Typical instances of such manifolds are obtained as quotients of the hyperbolic 3-space $\mathbb{H}^3$ by discrete subgroups of the full isometry group of $\mathbb{H}^3$.

Consider the set of all complete hyperbolic 3-manifolds with a finite volume. Then the set of volumes is totally ordered; moreover there exist only finitely many different hyperbolic 3-manifolds with the same volume [46]. Thus the volume of an hyperbolic manifold (unlike what happens in the Euclidean and elliptic cases) can be considered as a topological invariant. Computer geometry is the branch of geometric topology devoted to the calculation of these invariants: it still exhibits many open problems interesting for a quantum-computational approach, but it is most intriguing that such a hard 'computational' approach raised discussions among mathematicians about the 'philosophical' question of the effectiveness of mathematical proofs.

At first sight the above remarks on hyperbolic volumes does not seem related to our central issue of coloured quantum invariant of 3-manifolds. However, this is not the case: a connection can be easily recognized by observing that most manifolds obtained by surgery on framed knots (links) in the 3-sphere can be endowed with hyperbolic metrics. Let us focus for example on 'hyperbolic knots', namely those knots which give rise by surgery to (finite volume) hyperbolic 3-manifolds: the 'volume conjecture' proposed by [47, 48] (see also the review [40] for extended versions) can be cast in this case in the form

$$2\pi \lim_{N \to \infty} \frac{\log|J_N(K)|}{N} = \mathrm{Vol}(S^3 \backslash K), \tag{20}$$

where K is a hyperbolic knot and the notation $J_N(K)$ stands for the N-coloured polynomial of K evaluated at $q = \exp(2\pi \mathrm{i}/N)$.

Note that all the quantum algorithms dealing with link polynomials are established for a fixed choice of the root of unity q appearing in the argument of the invariants, while the volume conjecture involves the analysis of the asymptotic behaviour of 'single-coloured' polynomial of the same knot for increasing values of the colouring itself. Recently, Aharonov and Arad [49] have addressed an asymptotic analysis ($k \to \infty$) for the Jones polynomial, still $\frac{1}{2}$-coloured. It would be interesting to explore the possibility of borrowing some of their techniques to test the conjecture (20) within the computational framework designed for coloured polynomials.

Acknowledgment

References

[1] Jones V F R 1985 *Bull. Am. Math. Soc.* **12** 103
[2] Aharonov D, Jones V and Landau Z 2005 A polynomial quantum algorithm for approximating the Jones polynomial *Preprint* quant-ph/0511096

[3]   Garnerone S Marzuoli A and Rasetti A 2006 Quantum automata, braid group and link polynomials *Preprint* quant-ph/0601169

[4]   Arnowitt R, Deser S and Misner C W 1962 The dynamics of general relativity *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley)

[5]   Adler R, Bazin M and Schiffer M 1965 *Introduction to General Relativity* (New York: McGraw-Hill)

[6]   Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)

[7]   Wheeler J A 1968 Superspace and the nature of quantum geometrodynamics *Battelle Rencontres'* ed C M DeWitt and J A Wheeler (New York: Benjamin)

[8]   Weinberg S 1995 *The Quantum Theory of Fields—Foundations* vol 1 (Cambridge: Cambridge University Press)

[9]   Hartle J B and Hawking S W 1983 *Phys. Rev.* D **28** 2960

[10] Regge T 1961 *Nuovo Cimento* **19** 558

[11] Ambjorn J, Carfora M and Marzuoli A 1997 *The Geometry of Dynamical Triangulations (Lecture Notes in Physics* vol 50*)* (Berlin: Springer)

[12] Ambjorn J, Durhuus B and Jonsson T 1997 *Quantum Geometry* (Cambridge: Cambridge University Press)

[13] Yang C N and Mills R 1954 *Phys. Rev.* **96** 191

[14] Jackiw R 1980 *Rev. Mod. Phys.* **52** 661

[15] Eguchi T, Gilkey P B and Hanson A J 1980 *Phys. Rep.* **66** 213

[16] Atiyah M F 1989 Topological quantum field theories *Publ. Math. IHES* **68** 175

[17] Birmingham D, Blau M, Rakowski M and Thomson G 1991 *Phys. Rep.* **209** 129

[18] Quinn F 1995 Lectures on axiomatic topological quantum field theories *Geometry and Quantum Field Theory* ed D S Freed *et al* (Providence RI: Amer. Math. Soc.)

[19] Witten E 1989 *Commun. Math. Phys.* **121** 351

[20] Carlip S 1998 *Quantum Gravity in 2+1 Dimensions* (Cambridge: Cambridge University Press)

[21] Reshetikhin N and Turaev V G 1991 *Invent. Math.* **103** 547

[22] Kirby R and Melvin P 1991 *Inventiones Mathematicae* **105** 473

[23] Lickorish W B R 1997 *An Introduction to Knot Theory* (New York: Springer)

[24] Guadagnini E 1993 *The Link Invariants of the Chern–Simons Field Theory* (Berlin: de Gruyter)

[25] Ramadevi P, Govindarajan T R and Kaul R K 1994 *Mod. Phys. Lett.* A **9** 3205

[26] Garnerone S, Marzuoli A and Rasetti M 2006 Quantum Knitting *Preprint* quant-ph/0606137

[27] Kaul R K 1994 *Commun. Math. Phys.* **162** 289

[28] Chomsky N 1956 *IRE Trans. Inform. Theory* **2** 113

[29] Hopcroft J E and Ullman J D 1979 *Introduction to Automata Theory, Languages and Computation* (Reading, MA: Addison-Wesley)

[30] Moore C and Crutchfield J P 2000 *Theor. Comput. Sci.* **37** 275

[31] Marzuoli A and Rasetti M 2002 *Phys. Lett.* A **306** 79

[32] Marzuoli A and Rasetti M 2005 *Ann. Phys.* **318** 345

[33] Birman J S and Brendle T E 2004 Braids: a survey *Preprint* math/0409205

[34] Birman J S 1974 *Braids, Links and Mapping Class Groups* (Princeton, NJ: Princeton University Press)

[35] Harrow A, Recht B and Chuang I L 2002 *J. Math. Phys.* **43** 4445

[36] Bordewich M, Freedman M, Lovasz L and Welsh D 2005 Approximate counting and quantum computation *Comb. Probab. Comput.* **14** 737

[37] Wocjan P and Yard J 2006 The Jones polynomial: quantum algorithms and applications in quantum complexity theory *Preprint* quant-ph/0603069

[38] Dehornoy P 2004 The group of parenthesized braids *Preprint* math/0407097

[39] Bar-Natan D 1997 Non-associative tangles *Geometric Topology* ed W H Kazez *et al* (Providence, RI: American Mathematical Society) pp 139–83

[40] Ohtsuki T (ed) 2002 Problems on invariants of knots and 3-manifolds *RIMS Geom. Topol. Monogr.* **4** 377

[41] Menasco W and Thistlethwaite (ed) 2005 *Handbook of Knot Theory* (Amsterdam: Elsevier)

[42] Thurston W P 1997 *Three-Dimensional Geometry and Topology* vol 1 (Princeton: Princeton University Press)

[43] Lloyd S 2005 A Theory of quantum gravity based on quantum computation *Preprint* quant-ph/0501135

[44] Kauffman L H and Lomonaco S J 2006 Q-deformed spin networks, knot polynomials and anyonic topological quantum computation *Preprint* quant-ph/0606114

[45] Fomenko A T and Kunii T L 1997 *Topological Modeling for Visualization* (Tokyo: Springer)

[46] Gromov M 1981 *Hyperbolic manifolds (Lect. Notes in Math* vol 842*)* (Berlin: Springer) p 40

[47] Kashaev R M 1995 *Mod. Phys. Lett.* A **10** 1409

[48] Murakami H and Murakami J 2001 *Acta Math.* **186** 85

[49] Aharonov D and Arad I 2006 The BPQ-hardness of approximating the Jones polynomial *Preprint* quant-ph/0605181